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HARMONIC MAPS BETWEEN THE GROUP OF AUTOMORPHISMS OF THE QUATERNION ALGEBRA

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Abstract. In this paper, let Q be the real quaternion algebra which consists of all quaternionic numbers, and let G be the Lie group of all automorphisms of the algebra Q. Assume that g is an arbitrary given left invariant Riemannian metric on the Lie group G. Then, we obtain a necessary and sufficient condition for an automorphism of the group G to be harmonic.

1. Introduction

A harmonic map ϕ of a compact Riemannian manifold (M, g) into another Riemannian manifold (N, h) is a critical point of the energy functional ([1, 7])

$$E(\phi) := \int_M e(\phi)\ v_g, \tag{1.1}$$

where $e(\phi) := \frac{1}{2}h(d\phi, d\phi)$. To construct harmonic maps between two Riemannian manifolds is a very important topic in the study on the theory of harmonic maps.

It is well known that every inner automorphism of a compact connected Lie group G onto itself is both isometric and harmonic with respect to a bi-invariant Riemannian metric g_o on G. The present author ([3]) completely classified inner automorphisms of $(SU(2), g)$ with an arbitrary given left invariant Riemannian metric g onto $(SU(2), g)$ which are harmonic.

In this paper, let Q be the real quaternion algebra which consists of all quaternionic numbers, and let G be the Lie group of all automorphisms

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of the algebra Q. Then we obtain the fact that the group G may be regarded as $SO(3)$ (cf. Proposition 2.1). And then, we get a necessary condition (cf. Theorem 2.3) for a group homomorphism $\phi : G \longrightarrow G$ to be a group automorphism of G. Moreover, using Sugahara's Lemma (cf. Lemma 2.2), we obtain a necessary and sufficient condition (cf. Theorem 3.1) for a group homomorphism $\phi : G \longrightarrow G$ to be a group automorphism which is harmonic.

2. The Lie group of all automorphisms of the quaternion algebra

2.1. Let Q be a real 4-dimensional algebra such that Q has a basis $\{1, i, j, k\}$ as a real vector space and the product operation is subject to the following relations :

(i) associativity ;

(ii) $ri = ir, rj = jr, rk = kr$ for all $r \in R$;

(iii) $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$.

The algebra Q is said to be the *real quaternion algebra* ([2]).

In general, if a bijective map f of an algebra $\mathcal{U}$ onto $\mathcal{U}$ have the properties that

$$f(\lambda x) = \lambda f(x),\ f(x+y) = f(x) + f(y) \text{ and } f(xy) = f(x)f(y) \tag{2.1}$$

for all $\lambda \in R$ and $x,\ y \in \mathcal{U}$, then f is said to be an *automorphism* of $\mathcal{U}$ onto $\mathcal{U}$.

Now, putting $1 =: e_1, i =: e_2, j =: e_3, k =: e_4$ and $e_ie_j =: \sum_{k=1}^{4} \alpha_{ij}{}^k e_k$ in Q, we get

$$\begin{cases} \alpha_{11}{}^1 = \alpha_{12}{}^2 = \alpha_{13}{}^3 = \alpha_{14}{}^4 = \alpha_{21}{}^2 = \alpha_{31}{}^3 = \alpha_{41}{}^4 = 1, \\ \alpha_{22}{}^2 = \alpha_{33}{}^3 = \alpha_{44}{}^4 = -1, \\ \alpha_{23}{}^4 = -\alpha_{24}{}^3 = -\alpha_{32}{}^4 = \alpha_{34}{}^2 = \alpha_{42}{}^3 = -\alpha_{43}{}^2 = 1, \\ \alpha_{ij}{}^k = 0, \end{cases} \tag{2.2}$$

Let a be an R-linear map of Q onto Q which is defined by

$$\begin{aligned} a(e_1\ e_2\ e_3\ e_4) &:= (a(e_1)\ a(e_2)\ a(e_3)\ a(e_4)) \\ &= (e_1\ e_2\ e_3\ e_4)\ (a_j{}^i)_{i,j}, \end{aligned} \tag{2.3}$$

where $(a_j{}^i)_{i,j} \in GL_4(R)$. Then we have

$$a(e_i\ e_j) = \sum_{k,m=1}^{4} \alpha_{ij}{}^k a_k{}^m e_m,$$
$$a(e_i)\ a(e_j) = \sum_{k,l,m=1}^{4} a_i{}^k a_j{}^l \alpha_{kl}{}^m e_m. \tag{2.4}$$

By virtue of (2.4), the linear transformation a is an automorphism of the algebra Q if and only if

$$(a_j{}^i)_{i,j} \in GL_4(R), \text{ and}$$
$$\sum_{k=1}^{4} \alpha_{ij}{}^k a_k{}^m = \sum_{k,l=1}^{4} a_i{}^k a_j{}^l \alpha_{kl}{}^m, \quad (i,j,m = 1,2,3,4). \tag{2.5}$$

Moreover, under the product operation, Q is a noncommutative division ring, and the multiplicative inverse of $\sum_{i=1}^{4} \lambda_i e_i$ $(\in Q)$ is given by ([2])

$$\left(\sum_{i=1}^{4} \lambda_i e_i\right)^{-1} = (\lambda_1/d)e_1 - \sum_{i=2}^{4} (\lambda_i/d)e_i, \tag{2.6}$$

where each $\lambda_i \in R$ and $d := \sum_{i=1}^{4} \lambda_i{}^2$. To obtain a necessary and sufficient condition for an R-linear map $a : Q \to Q$ to be an automorphism of the algebra Q, using (2.2)-(2.6), we prepare step by step as follows.

First of all, $a(e_1) = a(e_1)\ a(e_1)$ holds if and only if

$$a_1{}^1 = 1 \text{ and } a_1{}^l = 0, \quad (l = 2,3,4). \tag{2.7}$$

Assume (2.7) holds. Then, for each $i \in \{2,,3,4\}$, $a(e_i\ e_i) = a(e_i)\ a(e_i)$ holds if and only if

$$a_i{}^1 = 0 \text{ and } \sum_{l=1}^{4} (a_i{}^l)^2 = 1. \tag{2.8}$$

Assume (2.7) and (2.8) hold. Then, for mutually different i, j $(i, j = 2,3,4)$,

$$a(e_ie_j) = a(e_i)a(e_j) = -a(e_je_i) = -a(e_j)a(e_i)$$

holds if and only if

$$\sum_{l=1}^{4} a_i{}^l a_j{}^l = 0. \tag{2.9}$$

Finally, assume (2.7), (2.8) and (2.9) hold. Then for the R-linear map a,

$$\begin{cases} a(e_2\ e_3) = a(e_2)\ a(e_3) = a(e_4), \\ a(e_3\ e_4) = a(e_3)\ a(e_4) = a(e_2), \\ a(e_4\ e_2) = a(e_4)\ a(e_2) = a(e_3) \end{cases}$$

are hold if and only if the R-linear map a is of the form

$$a \equiv (a_i{}^j)_{i,j} = \begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix} \in O(4) \text{ such that} \tag{2.10}$$

$$B = (b_{ij})_{i,j=1,2,3} = \begin{bmatrix} \triangle_{11} & \triangle_{12} & \triangle_{13} \\ \triangle_{21} & \triangle_{22} & \triangle_{23} \\ \triangle_{31} & \triangle_{32} & \triangle_{33} \end{bmatrix},$$

where $O(4)$ is the real orthogonal group of order 4, and $\triangle_{ij}$ is the (i,j)-cofactor of the matrix B. From (2.10), we obtain the fact that $detB = 1$, and so an automorphism a of Q is defined by

$$a(e_1\ e_2\ e_3\ e_4) = (e_1\ e_2\ e_3\ e_4) \begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix}, \tag{2.11}$$

$$B = (b_{ij})_{i,j} \in SO(3).$$

Thus, we have

PROPOSITION 2.1. *The Lie group G of all automorphisms of the real quaternion algebra Q is given as follows:*

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \mid B \in SO(3) \right\} = SO(3).$$

2.2. In this subsection, we introduce Sugahara's Lemma for later use.

Let B be the Killing form of the Lie algebra $\mathfrak{so}(3)$ of $SO(3)$. Since B is negative definite, $-B/2 =:<\ ,\ >_0$ defines a left invariant Riemannian metric on $SO(3)$, which we denote by g_0. The following Lemma is known ([8, Lemma 1.1]):

LEMMA 2.2. *Let g be a left invariant Riemannian metric and $<\ ,\ >$ an inner product defined by $< X, Y >:= g_e(X_e, Y_e)$, where $X, Y \in \mathfrak{so}(3)$ and e is the identity matrix of $SO(3)$. Then there exists an orthonormal basis (X_1, X_2, X_3) of $\mathfrak{so}(3)$ with respect to $<\ ,\ >_0 (:= -B/2)$ such that*

$$\begin{cases} [X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2, \\ < X_i, X_j >= \delta_{ij} a_i, \end{cases} \tag{2.12}$$

where a_i, $(1 \leq i \leq 3)$, are positive real numbers determined by the given left invariant Riemannian metric g of $SO(3)$.

2.3. Let H, $H^{'}$ be connected Lie groups and $\mathfrak{h}$, $\mathfrak{h}^{'}$ their Lie algebras, respectively. In general, if H is simply connected, then for a homomorphism α of $\mathfrak{h}$ onto $\mathfrak{h}^{'}$, there is a homomorphism φ of H onto $H^{'}$ such that $\varphi_* = \alpha$ ([6: Theorem 92 and, Examples 98 and 99]).

Let ϕ be an isomorphism of $SO(3)$. Then, $\phi_* : \mathfrak{so}(3) \longrightarrow \mathfrak{so}(3)$ is expressed by

$$\phi_*(X_1, X_2, X_3) = (X_1, X_2, X_3)(\phi^i_j)_{i,j}, \tag{2.13}$$

where $(\phi^i_j)_{i,j} \in GL_3(R)$ and (X_1, X_2, X_3) is the orthonormal basis of $\mathfrak{so}(3)$ with respect to $< \ , \ >_0 := -B/2$ which is appeared in Lemma 2.2. From (2.12), (2.13) and

$$\begin{cases} \phi_*([\ X_1,\ X_2\ ]) &= [\phi_* X_1,\ \phi_* X_2\ ] = \phi_*(X_3), \\ \phi_*([\ X_2,\ X_3\ ]) &= [\phi_* X_2,\ \phi_* X_3\ ] = \phi_*(X_1), \\ \phi_*([\ X_3,\ X_1\ ]) &= [\phi_* X_1,\ \phi_* X_2\ ] = \phi_*(X_2), \end{cases}$$

we obtain

$$\phi^i_j = \triangle_{ij}, \quad (i, j = 1, 2, 3), \tag{2.14}$$

where $\triangle_{ij}$ is the (i, j) -cofactor of the matrix $(\phi^i_j)_{i,j}$.

Summing up, we obtain from Proposition 2.1 and (2.14)

Theorem 2.3. *Let G be the Lie group of all automorphisms of the quaternion algebra, and let ϕ be a group isomorphism of $G(= SO(3))$. Let $(X_1,\ X_2,\ X_3)$ be the orthonormal basis of $(\mathfrak{so}(3), < \ , \ >_0)$ which is appeared in Lemma 2.2, and $\phi_*(X_1, X_2, X_3) =: (X_1, X_2, X_3)(\phi^i_j)_{i,j}$ and $(\phi^i_j)_{i,j} =: \Phi$. Then, $\Phi = det(\Phi)({}^t\Phi)^{-1}$, where ${}^t\Phi$ is the transpose of the matrix Φ.*

3. Harmonic automorphisms between the Lie group of all automorphisms of the quaternion algebra

3.1. Let ϕ be a C^∞-map of an m-dimensional compact Riemannian manifold (M, g) into another Riemannian manifold (N, h). If the map ϕ is a critical point of the energy functional

$$E(\psi) := \int_M e(\psi)\ v_g, \quad (\psi \in C^\infty(M, N)), \tag{3.1}$$

where $e(\psi) = \frac{1}{2}h(d\psi, d\psi)$, then ϕ is said to be a *harmonic map* ([1, 3, 7, 9]) of (M, g) into (N, h). Let ∇ and ${}^N\nabla$ be the Levi-Civita connection on (M, g) and (N, h), respectively. It is well known that a map $\phi : (M, g) \longrightarrow (N, h)$ is harmonic if and only if $\tau(\phi) = 0$ on M, where

$$\tau(\phi) := \sum_{i=1}^{m} ({}^N\nabla_{\phi_* e_i} \phi_* e_i - \phi_* \nabla_{e_i} e_i) \tag{3.2}$$

for an (locally defined) orthonormal frame $\{e_i\}_{i=1}^m$ on (M, g).

3.2. In this subsection, we get a necessary and sufficient condition for a group automorphism between the Lie group G of all automorphisms of the quaternion algebra to be harmonic. From now on, the group G may be regarded as $SO(3)$ (cf. Proposition 2.1).

Left invariant Riemannian metrics on $SO(3)$ are identified with the inner products of the Lie algebra $\mathfrak{so}(3)$. We fix an orthonormal basis $\{X_1, X_2, X_3\}$ of $\mathfrak{so}(3)$ with respect to $< \, , \, >_0 := -B/2$ with the property (2.12), and denote by $g_{(a_1,a_2,a_3)}$ the left invariant Riemannian metric on $SO(3)$ which is determined by positive real numbers a_1, a_2, a_3 in Lemma 2.2. If we put $Y_i := X_i/\sqrt{a_i}$, $(i = 1, 2, 3)$, then Y_1, Y_2, Y_3 is an orthonormal frame on $(SO(3), g_{(a_1,a_2,a_3)})$. From (2.12), we get

$$[Y_1, Y_2] = a_3 c^{-1} Y_3, \quad [Y_2, Y_3] = a_1 c^{-1} Y_1, \quad [Y_3, Y_1] = a_2 c^{-1} Y_2, \tag{3.3}$$

where $c := \sqrt{a_1 \; a_2 \; a_3}$.

In general, the Riemannian connection ∇ for the Riemannian metric g on a Riemannian manifold (M, g) is given by

$$\begin{aligned} 2 \, g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]) \end{aligned} \tag{3.4}$$

for $X, Y, Z \in \mathfrak{X}(M)$ ([4, 5]).

From now on, we denote by ∇ the Riemannian connection for the metric $g_{(a_1,a_2,a_3)}$ on $(SO(3), g_{(a_1,a_2,a_3)})$. By virtue of (3.3) and (3.4), we obtain

$$
(3.5)\qquad \begin{cases} \nabla_{Y_1} Y_1 = \nabla_{Y_2} Y_2 = \nabla_{Y_3} Y_3 = 0, \\ \nabla_{Y_1} Y_2 = (2c)^{-1}\ (a_2 + a_3 - a_1)\ Y_3, \\ \nabla_{Y_2} Y_1 = (2c)^{-1}\ (a_2 - a_3 - a_1)\ Y_3, \\ \nabla_{Y_2} Y_3 = (2c)^{-1}\ (a_1 - a_2 + a_3)\ Y_1, \\ \nabla_{Y_3} Y_2 = (2c)^{-1}\ (a_3 - a_1 - a_2)\ Y_1, \\ \nabla_{Y_3} Y_1 = (2c)^{-1}\ (a_1 + a_2 - a_3)\ Y_2, \\ \nabla_{Y_1} Y_3 = (2c)^{-1}\ (a_1 - a_2 - a_3)\ Y_2. \end{cases}
$$

Now, let ψ be a group automorphism of $SO(3)$. From (3.2) and (3.5), we get the fact that ψ is a harmonic map if and only if

$$
(3.6)\qquad \begin{aligned} \sum_{i=1}^{3}(a_2 - a_1)\ \bar{\psi}_i^1\ \bar{\psi}_i^2 &= \sum_{i=1}^{3}(a_3 - a_2)\ \bar{\psi}_i^2\ \bar{\psi}_i^3 \\ &= \sum_{i=1}^{3}(a_1 - a_3)\ \bar{\psi}_i^3\ \bar{\psi}_i^1 = 0, \end{aligned}
$$

where $\psi_* Y_j =: \sum_{k=1}^{3} \bar{\psi}_j^k Y_k$ $(j = 1, 2, 3)$. Putting $\psi_* X_j = \sum_{k=1}^{3} \psi_j^k X_k$ $(j = 1, 2, 3)$, we have from $Y_i = X_i/\sqrt{a_i}$ for each i

$$
(3.7)\qquad \bar{\psi}_j^i = (a_i/a_j)^{\frac{1}{2}}\ \psi_j^i, \quad (i, j = 1, 2, 3).
$$

By the help of (3.6) and (3.7), we get the fact that the group automorphism ψ is a harmonic map if and only if

$$
(3.8)\qquad \begin{aligned} \sum_{i=1}^{3}(a_2 - a_1) a_i \psi_i^1 \psi_i^2 &= \sum_{i=1}^{3}(a_3 - a_2) a_i \psi_i^2 \psi_i^3 \\ &= \sum_{i=1}^{3}(a_1 - a_3) \bar{\psi}_i^3 \bar{\psi}_i^1 = 0. \end{aligned}
$$

By virtue of (3.8), Proposition 2.1 and Theorem 2.3, we obtain

THEOREM 3.1. *Let G be the Lie group, with a left invariant Riemannian metric $g_{(a_1,a_2,a_3)}$, of all automorphisms of the quaternion algebra, and let ψ be a group automorphism of G. Let $\psi_*(X_1X_2X_3) =: (X_1X_2X_3)(\psi_j^i)_{i,j}$ and $(\psi_j^i)_{i,j} =: \Psi$. Then, the following statements are equivalent*:

(i) *ψ is an automorphism of G which is harmonic ;*

(ii) $\Psi = \det(\Psi)\ ({}^t\Psi)^{-1}$ *and*

$$\sum_{i=1}^{3}(a_2-a_1)a_i\psi_i^1\psi_i^2 = \sum_{i=1}^{3}(a_3-a_2)a_i\psi_i^2\psi_i^3 = \sum_{i=1}^{3}(a_1-a_3)a_i\psi_i^3\psi_i^1 = 0.$$

Using Theorem 3.1, we get the following ([3, Proposition 3.3])

EXAMPLE 3.2. *An inner automorphism $\psi = A_x,\ (x = exp\ rX_1,\ r \in R)$, of $(G = SO(3), g_{(a_1,a_2,a_3)})$ is harmonic if and only if*

$$a_2 = a_3 \quad or \quad r \in \{(n\pi)/2|\ n \text{ is an integer}\}, \tag{3.9}$$

where X_1 is the left invariant vector field on $SO(3)$ which is appeared in Lemma 2.2.

Proof. Using (2.12) and (3.3), we have

$$\begin{cases} \psi_*Y_1 = A_{x_*}(Y_1) = Ad(x)Y_1 = Y_1, \\ \psi_*Y_2 = A_{x_*}(Y_2) = Ad(x)Y_2 = \cos rY_2 + (\sqrt{a_3}/\sqrt{a_2})\sin rY_3, \\ \psi_*Y_3 = A_{x_*}(Y_3) = Ad(x)Y_3 = -(\sqrt{a_2}/\sqrt{a_3})\sin rY_2 + \cos rY_3, \end{cases} \tag{3.10}$$

where Ad is the adjoint representation of $SO(3)$. From (3.10), we get

$$\begin{cases} \bar\psi_1^1 = 1, \quad \bar\psi_1^2 = \bar\psi_1^3 = \bar\psi_2^1 = \bar\psi_3^1 = 0, \\ \bar\psi_2^2 = \bar\psi_3^3 = \cos r, \quad \bar\psi_2^3 = (\sqrt{a_3}/\sqrt{a_2})\sin r, \\ \bar\psi_3^2 = (-\sqrt{a_2}/\sqrt{a_3})\sin r. \end{cases} \tag{3.11}$$

By virtue of (3.7) and (3.11), we obtain

$$\begin{cases} \psi_1^1 = 1, \quad \psi_1^2 = \psi_1^3 = \psi_2^1 = \psi_3^1 = 0, \\ \psi_2^2 = \psi_3^3 = \cos r, \quad \psi_2^3 = \sin r, \quad \psi_3^2 = -\sin r. \end{cases} \tag{3.12}$$

From (3.12) and Theorem 3.1, we get the fact that $\psi(= A_{\exp rX_1})$ is harmonic if and only if

$$(a_3-a_2)^2\sin 2r = 0, \tag{3.13}$$

that is, $a_2 = a_3$ or $r \in \{(n\pi)/2|\ n\ is\ an\ integer\}$. □

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